Exercise 53

How many tangent lines to the curve $y = x/(x+1)$ pass through the point $(1, 2)$? At which points do these tangent lines touch the curve?

Solution

The equation of any line that goes through $(1, 2)$ is

$$y - 2 = m(x - 1) \quad \rightarrow \quad y = mx + 2 - m,$$

where m is the slope. For this line to be tangent to the given curve, the two functions representing them must be equal at some value of x.

$$\frac{x}{x+1} = mx + 2 - m \tag{1}$$

In addition, their slopes must be equal at this value of x.

$$\frac{1}{(x+1)^2} = m \tag{2}$$

Equations (1) and (2) can be solved for the unknowns, m and x. Substitute the formula for m into equation (1).

$$\frac{x}{x+1} = \frac{1}{(x+1)^2}x + 2 - \frac{1}{(x+1)^2}$$

Multiply both sides by $(x+1)^2$.

$$x(x+1) = x + 2(x+1)^2 - 1$$

Expand both sides.

$$x^2 + x = x + 2x^2 + 4x + 2 - 1$$

Bring all terms to one side.

$$x^2 + 4x + 1 = 0$$

Solve for x.

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2(1)} = -2 \pm \sqrt{3}$$

Again, these values of x are where tangent lines (that go through $(1, 2)$) intersect the given curve. Plug them into the given function.

$$y(-2+\sqrt{3}) = \frac{-2+\sqrt{3}}{-2+\sqrt{3}+1} = \frac{-2+\sqrt{3}}{-1+\sqrt{3}} \cdot \frac{-1-\sqrt{3}}{-1-\sqrt{3}} = \frac{-1+\sqrt{3}}{-2} = \frac{1-\sqrt{3}}{2}$$

$$y(-2-\sqrt{3}) = \frac{-2-\sqrt{3}}{-2-\sqrt{3}+1} = \frac{-2-\sqrt{3}}{-1-\sqrt{3}} \cdot \frac{-1+\sqrt{3}}{-1+\sqrt{3}} = \frac{-1-\sqrt{3}}{-2} = \frac{1+\sqrt{3}}{2}$$

Therefore, the points that the (two) tangent lines touch the curve are

$$\left(-2+\sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \quad \text{and} \quad \left(-2-\sqrt{3}, \frac{1+\sqrt{3}}{2}\right).$$